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A HIGHER-ORDER PLATE THEORY WITH IDEAL FINITE ELEMENT SUITABILITY

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ABSTRACT

A variationally consistent tenth-order displacement theory of stretching and bending of orthotropic elastic plates is proposed which lends itself perfectly to finite element formulations based upon C^0 and C^{-1}-continuous displacement approximations. The deformations due to all strain and stress components are accounted. The theory is derived from three-dimensional elasticity via the principle of virtual work by expanding the displacement components with respect to the thickness coordinate by means of Legendre polynomials, where the transverse displacement is of a special parabolic form while the inplane displacements are linear. The issues of thickness-expansion related inconsistencies in the transverse shear strains and the transverse normal stress are resolved in a rational fashion. The resulting parabolic shear strains incorporate Reissner's shear correction factor, while the transverse normal strain varies cubically across the plate thickness. The variational principle yields seven equations of motion and exclusively Poisson-type edge boundary conditions. A qualitative assessment of the theory is carried out for the problem of static equilibrium involving an infinite plate under a sinusoidal normal pressure. Pertinent issues on the particular suitability of the theory for the development of efficient displacement plate elements are discussed.

CONTENTS

INTRODUCTION

In many design-critical regions in plate structures, such as those involving highly stressed areas near holes, cutouts and impact, bolted and adhesive joints, etc., deformations due to transverse shear and transverse normal strains and stresses can significantly contribute to the three-dimensional stress state. These effects are especially pronounced in composite laminates that are inherently compliant in the transverse shear and transverse normal modes of deformation. To adequately model such critical plate regions with finite elements, three-dimensional elements are commonly used, usually at a very high computational cost. Alternatively, from the standpoint of computational efficiency, and in order to avoid or minimize costly substructuring with three-dimensional elements, it would be most desirable to discretize with accurate plate elements that incorporate all strain and stress effects.

Many effective plate finite elements developed to date evolved from a shear-deformable displacement theory originally derived for homogeneous isotropic plates by Mindlin.[1] The theory lends itself to the development of simple and efficient elements by virtue of the inherent C^0-continuity prescribed for the approximation of the plate displacement variables. An earlier plate theory of Reissner[2-4] is based on stress approximations including the transverse normal stress component (which is neglected in the Mindlin theory) and is somewhat more accurate as far as the stresses are concerned. Both of the Mindlin and Reissner theories rely upon weighted-average displacement variables which constitute linear variations across the thickness for the inplane displacements and a uniform transverse displacement. From a perspective of composite laminate analysis, the displacement-based theories are generally preferred, allowing for discontinuous normal stresses which occur naturally in composite laminates (e.g., refer to Reference 5).

In recent years, several **higher-order shear deformable** displacement plate theories have been introduced (e.g., see References 6 and 7), in which the inplane displacements vary cubically across the plate thickness while the transverse displacement is uniform. While these theories appear to have a wider applicability range than their predecessors,[1-5] they possess a serious drawback from the finite element standpoint, the requirement of C^1 continuity which is known to inhibit development of simple and efficient elements (e.g., see discussions in Reference 8).

Higher-order plate theories,[9-16] which in addition to transverse shear also include the effect of transverse normal straining, are intended to extend the range of applicability to design-critical situations such as those mentioned earlier. The interested reader may find insightful reviews of many notable higher-order theories in the papers by Lo, Christensen, and Wu[15] and Reissner.[17] Regrettably, these theories are rather unattractive from the finite element viewpoint for the following reasons: (i) an excessively large number of plate displacement variables (e.g., Lo et al.'s theory[15] possesses 11 kinematic variables for the problem of bending and stretching, whereas the Reissner and Mindlin theories involve only five such variables), (ii) natural edge-boundary conditions that include nonclassical quantities such as higher-order force and moment resultants[9-15] (in addition to the physically meaningful Poisson conditions including three prescribed edge forces, bending, and twisting moments), (iii) a lack of variational basis,[16] and (iv) a departure from the kinematically admissible evaluation of stress;[18] i.e., evaluation of stress components from equilibrium equations rather than (in a kinematically admissible manner) from constitutive relations.

The present development of a higher-order plate theory, incorporating all stress and strain effects, is motivated by finite element application, where a variational formulation is desired requiring, at most, C^0-continuous displacement fields with the least number of variables, and relying exclusively upon Poisson-type boundary conditions.

In the Kinematic Assumptions Section, we expand three Cartesian displacements in terms of a thickness coordinate using Legendre polynomials to simplify integration across the thickness. To obtain the simplest higher-order theory, we adopt the same expansion order as proposed by Hildebrand, Reissner, and Thomas;[9] linear and parabolic distributions for the inplane and transverse displacements, respectively. In contrast to Hildebrand et al.'s transverse displacement expansion in which the leading term is the midplane displacement, the leading term in the present expansion is Reissner's[4] weighted-average displacement. Here, as in Reference 9 the parabolic transverse displacement also involves two higher-order displacement components. The particular advantage of the present expansion strategy is manifested in finite element approximations, where all kinematic variables need not exceed C^0 continuity. In the Concluding Summary and Finite Element Suitability Section, we shall remark further on some important consequences of the present continuity requirements on finite element development.

In the Transverse Shear Field Consistency and Transverse Normal Field Consistency Sections, we delineate certain types of inconsistencies that arise in the thickness expansion for the transverse shear strains and transverse normal stress, and propose a novel approach which eradicates these inconsistencies in a rational manner. In the Governing Equations Section, we employ the principle of virtual work, which yields seven partial differential equations of motion and exclusively Poisson edge boundary conditions. In the special case of static equilibrium, the theory decouples into one of fourth-order inplane stretching, sixth-order transverse bending, and zero-order transverse stretching. Whereas the first two sets of equations correspond to Reissner's theory,[2-4] the last set includes two higher-order transverse displacements of zero-order (i.e., they possess no spatial derivatives) and first-order derivatives of the weighted-average variables. The solution to a plate boundary value problem simply involves solving the fourth-order inplane stretching and sixth-order transverse bending equations subject to Poisson boundary conditions (as in Reissner's theory), and then a straightforward evaluation of the two higher-order variables from the zero-order transverse stretching equations.

One remarkable aspect of the theory is the particularly accurate representation of all transverse stresses. Thus, the kinematically admissible transverse shear and normal stresses are shown to satisfy exactly the transverse equilibrium equation of elasticity, i.e., they are also statically admissible in the transverse equilibrium sense. The transverse shear stresses vary parabolically across the thickness while the transverse normal stress has a cubic distribution. All transverse stresses satisfy the prescribed tractions at the top and bottom plate faces.

In the Assessment of Theory Section, we examine the range of applicability of the theory via an analytic solution for an infinite isotropic plate under quasi-static sinusoidal normal pressure and compare results with the exact elasticity solution and several other plate theories.

Finally, in the Concluding Summary and Finite Element Suitability Section, we discuss the issues concerning utilization of this variational displacement theory within the finite element framework, and suggest ways of formulating displacement plate elements whose computational efficiency would be comparable to Reissner-Mindlin type elements.[8,19,20]

KINEMATIC ASSUMPTIONS

Consider an orthotropic, homogeneous elastic plate located in the three-dimensional Cartesian framework (x, y, z), where the x-y plane passes through the middle of the plate thickness $\xi = 0$, where $\xi = z/h \in [-1, 1]$ is a dimensionless thickness coordinate, and 2h is the plate thickness. The top (S^+) and bottom (S^-) plate surfaces are taken free of shear tractions and loaded by normal pressures

$$\tau_{iz}(\xi=1) = 0 \ (i=x,y), \quad \sigma_z(\xi=1) = q^+(x,y,t) \qquad \text{on } S^+ \tag{1}$$

$$\tau_{iz}(\xi=-1) = 0 \ (i=x,y), \quad \sigma_z(\xi=-1) = q^-(x,y,t) \qquad \text{on } S^- \tag{2}$$

where t denotes time. On a part, S_σ, of the cylindrical boundary surface S normal to the midplane S_m, a traction vector is prescribed:

$$\bar{T} = \{\bar{T}_x, \bar{T}_y, \bar{T}_z\} \text{ on } S_\sigma . \tag{3}$$

We assume the plate principal material directions to be coincidents with the Cartesian directions, and small displacements.

The simplest physically meaningful approximations of displacement components allowing for the transverse normal straining involve linear through-thickness expansions of the inplane displacements (as in classical and refined theories), and a parabolic approximation of the transverse displacement. This level of approximation was first discussed by Hildebrand et al.[9] For convenience, we expand the Cartesian displacement components by means of Legendre polynomials with u_z having a special form:

$$\begin{aligned} u_x(x,y,z,t) &= u(x,y,t) + hP_1(\xi)\theta_y(x,y,t), \\ u_y(x,y,z,t) &= v(x,y,t) + hP_1(\xi)\theta_x(x,y,t), \\ u_z(x,y,z,t) &= w(x,y,t) + P_1(\xi)w_1(x,y,t) + [P_0(\xi)/5+P_2(\xi)]w_2(x,y,t), \end{aligned} \tag{4}$$

where

$$P_n(\xi) = \frac{1}{2^n \, n!} d^n[(\xi^2-1)^n]/d\xi^n, \quad P_n(1)=1, \ (\xi = z/h \in [-1,1])$$
$$(\text{i.e., } P_0 = 1, \quad P_1 = \xi, \quad P_2 = (3\xi^2-1)/2, \quad P_3 = \xi(5\xi^2-3)/2 \ ) \tag{5}$$

are Legendre polynomials which, due to their orthogonality property,

$$\int_{-1}^{1} P_m P_n \, d\xi = \begin{cases} 0 & \text{if } m \neq n \\ 2/(2m+1) & \text{if } m=n \end{cases} \tag{6}$$

prove convenient whenever thickness integrations are performed.

The expansion coefficients u, v, θ_y, θ_x, and w in Equation 4 are weighted average kinematic variables defined here in accordance with Reissner.[4]

$$u = \frac{1}{2h}\int_{-h}^{h} u_x dz, \quad v = \frac{1}{2h}\int_{-h}^{h} u_y dz \tag{7}$$

$$\theta_x = \frac{3}{2h^3}\int_{-h}^{h} u_y z\, dz, \quad \theta_y = \frac{3}{2h^3}\int_{-h}^{h} u_x z\, dz \tag{8}$$

$$w = \frac{1}{2h}\int_{-h}^{h} u_z (P_0 - P_2)\, dz, \quad \{ P_0 - P_2 = 3(1 - \xi^2)/2 \}, \tag{9}$$

where u and v denote the midplane displacements along the x and y directions, respectively and θ_x and θ_y denote the normal rotations about the x and y axes, respectively. In Equation 4, w_1 and w_2 are the higher-order components of a transverse displacement, u_z, which varies parabolically across the thickness. Note that w is the weighted average transverse displacement, in contrast to other higher-order theories[9-15] employing the midplane displacement variable. As will be seen, the special expansion for u_z leads to exclusively Poisson-type edge boundary conditions which are naturally obtained from the virtual work principle.

The plate strain-displacement relations are obtained in the usual manner by introducing Equation 4 into the strain-displacement relations of three-dimensional elasticity:

$$\varepsilon_i = u_{i,i}, \qquad \gamma_{ij} = (u_{i,j} + u_{j,i}), \quad (i,j = x,y,z). \tag{10}$$

resulting in

$$\left.\begin{aligned} \varepsilon_x &= u_{,x} + h\xi\theta_{y,x}, \quad \varepsilon_y = v_{,y} + h\xi\theta_{x,y} \\ \gamma_{xy} &= u_{,y} + v_{,x} + h\xi(\theta_{y,y} + \theta_{x,x}) \end{aligned}\right] \tag{11a}$$

$$\left.\begin{aligned} \gamma_{xz} &= (\gamma_{xz^0} - \frac{3}{10} w_{2,x}) + \xi w_{1,x} + \frac{3}{2}\xi^2 w_{2,x} \\ \gamma_{yz} &= (\gamma_{yz^0} - \frac{3}{10} w_{2,y}) + \xi w_{1,y} + \frac{3}{2}\xi^2 w_{2,y} \\ \gamma_{xz_0} &= \theta_y + w_{,x}, \quad \gamma_{yz_0} = \theta_x + w_{,y} \end{aligned}\right] \tag{11b}$$

$$\varepsilon_z = (w_1 + 3\xi w_2)/h \tag{11c}$$

where a comma (,) denotes partial differentiation with respect to the spatial coordinates. To facilitate our subsequent discussion, we expressed the strains in Equations 11 directly in terms of the thickness coordinate ξ.

The complete Hooke's law for a homogeneous orthotropic plate may be expressed in matrix form as:

$$\sigma = C\,\varepsilon,$$

or

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ & C_{22} & C_{23} & 0 & 0 & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & \text{(Symm.)} & & & C_{55} & 0 \\ & & & & & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{bmatrix}. \tag{12}$$

TRANSVERSE SHEAR FIELD CONSISTENCY

From Equation 11b, it becomes evident that certain ξ-distribution inconsistencies are present in the expansions of γ_{xz} and γ_{yz}. Here, both linear and quadratic terms in ξ involve only w_1 and w_2, which are contributed from u_z. This is obviously the result of $u_{i,z}$ $(i = x, y)$ being uniform across the thickness, whereas $u_{z,i}$ vary parabolically. It is clear that to possess a consistent kinematic form, each ξ-polynomial coefficient must include contributions from both the inplane and transverse displacements, as implied in Equation 10.

A simple way of assessing the effect of the aforementioned inconsistency is to examine the nature of the transverse shear strains for a plate under the Kirchhoff constraint, i.e., $\gamma_{iz} \rightarrow 0$ $(i = x, y)$ with the plate thickness diminishing to zero. From each shear strain, there results three constraint equations:

$$\begin{aligned} &[(\gamma_{xz_0} - \tfrac{3}{10} w_{2,x}),\quad w_{1,x},\quad w_{2,x}\,] \rightarrow 0 \\ &[(\gamma_{yz_0} - \tfrac{3}{10} w_{2,y}),\quad w_{1,y},\quad w_{2,y}\,] \rightarrow 0\,. \end{aligned} \tag{13}$$

It can be shown that (e.g., see results in the Assessment of Theory Section):

$$\begin{aligned} \frac{w_{1,i}}{\gamma_{iz_0}} &= O(2h/L)^4 \\ \frac{w_{2,i}}{\gamma_{iz_0}} &= O(2h/L)^2, \quad (i = x, y) \end{aligned} \tag{13a}$$

where L denotes some characteristic plate span. Accounting for Equation 13a in Equation 13, it becomes apparent that, at least in the Kirchhoff regime, the resulting shear strains are uniform across the plate thickness:

$$\gamma_{iz} = \gamma_{iz^0} \to 0, \quad (i=x,y). \tag{14}$$

The direct implication of Equation 14 is that explicit conditions of zero shear tractions on the plate faces cannot be satisfied and, moreover, an ad hoc specification of a shear correction factor is necessary to achieve agreement with the Kirchhoff theory for the class of thin plate problems.[11,12]

To eradicate this type of kinematic inconsistency while retaining a parabolic distribution for u_z, the gradients $u_{i,z}$ (i = x, y) should naturally be represented by a complete quadratic polynomial in ξ, thus implying cubic distributions for u_x and u_y. Approximations of this type were employed by Lo et al.[15] (who extended Reissner's work[14] in which inplane modes of deformation were neglected) resulting in a complex twenty-second-order theory of limited practical usefulness.

In the present treatment, in order to resolve this difficulty while preserving the simplicity of Equation 4, we propose to replace the uniformly distributed gradients $u_{i,z}$ (i = x, y) (directly computed from Equations 4 and 10) with the field-consistent gradients $u^*_{i,z}$ which vary parabolically across the thickness:

$$u^*_{i,z} = \sum_{k=0}^{2} a_{ik}(x,y,t)\, P_k(\xi), \quad (i=x,y). \tag{15}$$

Henceforth, the superior asterisk (*) will denote field-consistent gradient and resulting stress quantities. The relations between a_{ik} and plate displacement variables are determined via two types of physically desirable constraint conditions:

The field-consistent transverse shear stresses are required to satisfy traction-free boundary conditions at the top and bottom plate faces:

$$\tau^*_{iz}\Big|_{(\xi=1,\ -1)} = 0 \quad (i=x,y). \tag{16}$$

Taking into account the constitutive relations, Equation 12, the above stress constraints give rise to the vanishing conditions upon the shear strains:

$$\gamma^*_{iz} = \{u^*_{i,z} + u_{z,i}\}\Big|_{(\xi=1,\ -1)} = 0 \quad (i=x,y). \tag{16a}$$

Furthermore, the field-consistent shear strains are required to be equivalent in the mean to their field-inconsistent counterparts from Equation 11b:

$$\min \int_{-h}^{h} (\gamma_{iz}^{*} - \gamma_{iz})^2 \, dz, \quad (i=x,y). \tag{17}$$

Equations 16a and 17 are solved for a_{ik} (i = x, y; k = 0, 1, 2), giving rise to the consistent shear strains:

$$\gamma_{xz}^{*} = k^2(P_3 - P_2)(w_{,x} + \theta_y)$$
$$\gamma_{yz}^{*} = k^2(P_3 - P_2)(w_{,y} + \theta_x) \tag{18}$$

in which k^2 = 5/6 emerged from this derivation and is coincident with Reissner's shear correction factor.

TRANSVERSE NORMAL FIELD CONSISTENCY

A stress type of field inconsistency can be detected in the transverse equilibrium equation of the three-dimensional elasticity:

$$\tau_{xz,x} + \tau_{yz,y} + \sigma_{z,z} = 0 \quad \text{(ignoring the body force)}. \tag{19}$$

Here, owing to the parabolic ξ-distribution of τ_{iz} (i = x, y), σ_z must be cubic in ξ: however, assumptions, Equation 4, can only yield σ which is linear. To overcome this discrepancy, we replace $u_{z,z}$ (computed directly from Equations 4 and 10) with the field-consistent gradient $u^{*}_{z,z}$, having a cubic ξ-distribution across the thickness:

$$\varepsilon_z^{*} = u_{z,z}^{*} = \sum_{k=0}^{3} b_k(x,y,t)\, P_k(\xi) \tag{20}$$

where b_k are determined from the following constraint equations:

The homogeneous boundary conditions on the through-thickness gradient of transverse normal stress component:

$$\frac{\partial}{\partial z}(\sigma_z^{*}) \Big|_{(\xi=1,-1)} = 0, \tag{21}$$

These conditions result from Equation 19 when taking the top and bottom plate faces to be free of shear tractions.

The field-consistent transverse normal strain is required to be equivalent in the mean to its field-inconsistent counterpart of Equation 11c:

$$\min \int_{-h}^{h} (\varepsilon_z^{*} - \varepsilon_z)^2 \, dz. \tag{22}$$

From Equations 21 and 22, there results four equations in terms of b_k yielding a cubic transverse normal strain of the form:

$$\varepsilon_z^* = \frac{1}{h}\{w_1 + k_z^2 [(6P_2 - P_3)w_2 - \frac{h^2}{42C_{33}}(14P_3 + P_2)(C_{31}\theta_{y,x} + C_{32}\theta_{x,y})]\}$$

with (23)

$$k_z^2 = 42/85$$

In the remaining part of the formulation we shall exclusively deal with the field-consistent strains and stresses and, hence, we shall omit the superior asterisk which was previously used to distinguish these quantities.

GOVERNING EQUATIONS

We derive the equations of motion together with the natural boundary conditions via the principle of virtual work incorporating the field-consistent strains and stresses:

$$\int_{t_1}^{t_2} \Big\{ \iiint_V (\sigma_x\,\delta\varepsilon_x + \sigma_y\,\delta\varepsilon_y + \sigma_z\,\delta\varepsilon_z + \tau_{xy}\delta\gamma_{xy} + \tau_{xz}\delta\gamma_{xz} + \tau_{yz}\delta\gamma_{yz})\,dxdydz$$

$$-\delta \iiint_V \tfrac{1}{2}\rho\,(\dot u_x^2 + \dot u_y^2 + \dot u_z^2)\,dxdydz \qquad (24)$$

$$-\iint_{S^+} q^+\delta u_z dxdy - \iint_{S^-} q^-\delta u_z dxdy - \iint_{S_\sigma} (\bar T_x\delta u_x + \bar T_y\delta u_y + \bar T_z\delta u_z)\,dsdz \Big\}\,dt = 0$$

where s is a coordinate tangent to the boundary of S_m, ρ is the mass density, and the superior dot ($\dot{}$) denotes differentiation with respect to time.

Consistent with the parabolic distribution of the shear strains, Equation 18, the prescribed edge shear traction is assumed to vary parabolically as well, i.e., $\bar T_z = (P_0 - P_2)\,\tilde T_z(s)$. Introducing Equations 11a, 18 and 23 into Equation 24, and integrating with respect to z, we find that certain integrals vanish identically

$$\int_{-h}^{h} \bar T_z P_1 dz = \int_{-h}^{h} \bar T_z (P_0/3 - P_2)\,dz = \int_{-h}^{h} \sigma_z\,(P_1 + 14P_3)\,dz = 0, \qquad (25a)$$

yielding a two-dimensional virtual work statement of the form:

$$\int_{t_1}^{t_2} \Big[\iint_{S_m} \{ N_x\delta u_{,x} + N_y\delta v_{,y} + N_z\delta(w_1/h) + N_{xy}(\delta u_{,y} + \delta v_{,x})$$

$$+ M_x\delta\theta_{y,x} + M_y\delta\theta_{x,y} + M_z\delta(w_2/h) + M_{xy}(\delta\theta_{x,x} + \delta\theta_{y,y}) \qquad (25b)$$

$$+ Q_x(\delta w_{,x} + \delta\theta_y) + Q_y(\delta w_{,y} + \delta\theta_x) - q_1(\delta w + \delta w_2/k^2) - q_2\delta w_1$$

$$- m(\dot{u}\delta\dot{u} + \dot{v}\delta\dot{v}) - I_m(\dot{\theta}_y\delta\dot{\theta}_y + \dot{\theta}_x\delta\dot{\theta}_x) - m(\dot{w} + \dot{w}_2/5)\delta\dot{w} - \frac{m}{3}\dot{w}_1\delta\dot{w}_1$$

$$- \frac{m}{5}(\dot{w} + 2\dot{w}_2)\delta\dot{w}_2 \Big\} dxdy$$

$$+ \bar{Q}_{z_1}\delta w_1 + \bar{Q}_{z_2}\delta w_2 \Big\} ds \Bigg] dt = 0$$

$$- \oint_{C_\sigma} \Big\{ \bar{N}_{xn}\delta u + \bar{N}_{yn}\delta v + \bar{M}_{xn}\delta\theta_y + \bar{M}_{yn}\delta\theta_x + \bar{Q}_{zn}\delta w$$

Performing appropriate variations and integration by parts yields:

$$\int_{t_1}^{t_2} \Bigg[\iint_{S_m} \Big\{ [m\ddot{u} - N_{x,x} - N_{xy,y}]\,\delta u + [m\ddot{v} - N_{y,y} - N_{xy,x}]\,\delta v$$

$$+ [I_m\ddot{\theta}_y - M_{x,x} - M_{xy,y} + Q_x]\,\delta\theta_y + [I_m\ddot{\theta}_x - M_{y,y} - M_{xy,x} + Q_y]\,\delta\theta_x$$

$$+ [m(\ddot{w} + \frac{1}{5}\ddot{w}_2) - Q_{x,x} - Q_{y,y} - q_1]\,\delta w$$

$$+ [\frac{m}{3}\ddot{w}_1 + N_z/h - q_2]\,\delta w_1 \tag{26}$$

$$+ [\frac{m}{5}(\ddot{w} + 2\ddot{w}_2) + M_z/h - q_1/k^2]\,\delta w_2 \Big\} dxdy$$

$$+ \oint_{C_\sigma} \Big\{ (N_{xn} - \bar{N}_{xn})\,\delta u + (N_{yn} - \bar{N}_{yn})\,\delta v + (M_{xn} - \bar{M}_{xn})\,\delta\theta_y + (M_{yn} - \bar{M}_{yn})\,\delta\theta_x$$

$$+ (Q_{zn} - \bar{Q}_{zn})\,\delta w \Big\} ds$$

$$+ \oint_{C_u} \Big\{ N_{xn}\delta u + N_{yn}\delta v + M_{xn}\delta\theta_y + M_{yn}\delta\theta_x + Q_{zn}\delta w \Big\} ds \Bigg] dt = 0$$

where C_σ and C_u are parts of the boundary surrounding S_m where the tractions and displacements are prescribed, respectively; and where the inertial and stress resultants are defined as:

$$(m, I_m) = \int_{-h}^{h} \rho(1, z^2)\,dz, \quad q_1 = q^+ - q^-, \quad q_2 = q^+ + q^-,$$

$$(N_x, N_y, N_{xy}) = \int_{-h}^{h} (\sigma_x, \sigma_y, \tau_{xy})\, dz,$$

$$(M_x, M_y, M_{xy}) = \int_{-h}^{h} (\sigma_x, \sigma_y, \tau_{xy}) z\, dz,$$

$$(Q_x, Q_y) = \int_{-h}^{h} k^2(P_0 - P_2)(\tau_{xz}, \tau_{yz})\, dz, \tag{27}$$

$$N_z = \int_{-h}^{h} \sigma_z\, dz, \quad M_z = \int_{-h}^{h} \sigma_z\, k_z^2\, (6P_1 - P_3)\, dz,$$

$$(\bar{N}_{xn}, \bar{N}_{yn}) = \int_{-h}^{h} (\bar{T}_x, \bar{T}_y)\, dz, \quad (\bar{M}_{xn}, \bar{M}_{yn}) = \int_{-h}^{h} (\bar{T}_x, \bar{T}_y) z\, dz,$$

$$\bar{Q}_{zn} = \int_{-h}^{h} \bar{T}_z\, dz,$$

$$N_{xn} = N_x \ell + N_{xy} m, \quad N_{yn} = N_{xy} \ell + N_y m,$$

$$M_{xn} = M_x \ell + M_{xy} m, \quad M_{yn} = M_{xy} \ell + M_y m, \quad Q_{zn} = Q_x \ell + Q_y m,$$

$$\ell = \cos(x,n), \quad m = \cos(y,n),$$

where n is an outward normal to the boundary of S_m.

Thus, the principle provides seven equations of motion:

$$(\delta u): \quad N_{x,x} + N_{xy,y} = m\ddot{u} \tag{28a}$$

$$(\delta v): \quad N_{xy,x} + N_{y,y} = m\ddot{v} \tag{28b}$$

$$(\delta\theta_y): \quad M_{x,x} + M_{xy,y} - Q_x = I_m \ddot{\theta}_y \tag{28c}$$

$$(\delta\theta_x): \quad M_{xy,x} + M_{y,y} - Q_y = I_m \ddot{\theta}_x \tag{28d}$$

$$(\delta w): \quad Q_{x,x} + Q_{y,y} + q_1 = \frac{m}{5}(5\ddot{w} + \ddot{w}_2) \tag{28e}$$

$$(\delta w_1) \quad -N_z/h + q_2 = \frac{m}{3}\ddot{w}_1 \tag{28f}$$

$$(\delta w_2): \quad -M_z/h + q_2/k^2 = \frac{m}{5}(\ddot{w} + 2\ddot{w}_2) \tag{28g}$$

five stress boundary conditions on C_σ:

$$N_{xn} = \bar{N}_{xn}, \quad N_{yn} = \bar{N}_{yn}, \quad M_{xn} = \bar{M}_{xn}, \quad M_{yn} = \bar{M}_{yn}, \quad Q_{zn} = \bar{Q}_{zn}, \tag{29a}$$

and five displacement boundary conditions on C_u:

$$u = \bar{u}, \quad v = \bar{v}, \quad \theta_y = \bar{\theta}_y, \quad \theta_x = \bar{\theta}_x, \quad w = \bar{w}. \tag{29b}$$

Note that Equation 29 represents Poisson-type boundary conditions.

Invoking Hooke's law for an orthotropic material, Equation 12, yields the plate stress-displacement relations of the form:

$$N_x = \bar{A}_{11}u,_x + \bar{A}_{12}v,_y + \bar{A}_{13}\frac{w_1}{h}, \tag{30a}$$

$$N_y = \bar{A}_{12}u,_x + \bar{A}_{22}v,_y + \bar{A}_{23}\frac{w_1}{h}, \tag{30b}$$

$$N_{xy} = A_{66}(u,_y + v,_x), \tag{30c}$$

$$M_x = \bar{D}_{11}\theta_y,_x + \bar{D}_{12}\theta_x,_y + \bar{D}_{13}\frac{w_2}{h}, \tag{30d}$$

$$M_y = \bar{D}_{12}\theta_y,_x + \bar{D}_{22}\theta_x,_y + \bar{D}_{23}\frac{w_2}{h}, \tag{30e}$$

$$M_{xy} = D_{66}(\theta_x,_x + \theta_y,_y), \tag{30f}$$

$$Q_x = G_{55}(w,_x + \theta_y), \tag{30g}$$

$$Q_y = G_{44}(w,_y + \theta_x), \tag{30h}$$

$$N_z = \bar{A}_{13}u,_x + \bar{A}_{23}v,_y + \bar{A}_{33}\frac{w_1}{h}, \tag{30i}$$

$$M_z = \bar{D}_{13}\theta_y,_x + \bar{D}_{23}\theta_x,_y + \bar{D}_{33}\frac{w_2}{h}, \tag{30j}$$

where

$$\begin{aligned}
&\bar{A}_{ij} = (2h)\,C_{ij}, \quad (i,j=1,2,3),\\
&\bar{D}_{ij} = (2h^3/3)\left[C_{ij} - \frac{C_{i3}C_{j3}}{85\,C_{33}}\right] \quad (i,j=1,2),\\
&\bar{D}_{i3} = k_z^2\,(2h)^2\,C_{i3} \quad (i=1,2),\\
&\bar{D}_{33} = k_z^2\,(12h)\,C_{33},\\
&A_{66} = (2h)\,C_{66}, \quad D_{66} = (2h^3/3)\,C_{66},\\
&G_{ii} = k^2\,(2h)\,C_{ii} \quad (i=4,5),
\end{aligned} \tag{31}$$

Special Case: Plate in Equilibrium

It is of significance to consider the case of a quasi-static loading, leading to the vanishing of the inertial terms, in which case Equations 28a through 28g become equilibrium equations. Substituting Equations 30i and 30j into Equations 28f and 28g, respectively, results in the solution for w_1 and w_2:

$$\frac{w_1}{h} = \frac{1}{\bar{A}_{33}}\left[q_2 h - \bar{A}_{13}u,_x - \bar{A}_{23}v,_y \right] \tag{32a}$$

$$\frac{w_2}{h} = \frac{1}{\bar{D}_{33}}\left[q_1 \frac{h}{k^2} - \bar{D}_{13}\theta_y,_x - \bar{D}_{23}\theta_x,_y \right]. \tag{32b}$$

Introducing Equation 32 into Equation 30 yields the expressions for N_x, N_y, M_x, and M_y:

$$\left.\begin{aligned} N_x &= A_{11}u,_x + A_{12}v,_y + \frac{C_{13}q_2h}{C_{33}} \\ N_y &= A_{12}u,_x + A_{22}v,_y + \frac{C_{23}q_2h}{C_{33}} \end{aligned}\right] \tag{33a}$$

$$\left.\begin{aligned} M_x &= D_{11}\theta_y,_x + D_{12}\theta_x,_y + \frac{2C_{13}q_1h^2}{5C_{33}} \\ M_y &= D_{12}\theta_y,_x + D_{22}\theta_x,_y + \frac{2C_{23}q_1h^2}{5C_{33}} \end{aligned}\right] \tag{33b}$$

where

$$\begin{aligned} A_{ij} &= (2h)\left[C_{ij} - \frac{C_{i3}C_{j3}}{C_{33}}\right], \\ D_{ij} &= (2h^3/3)\left[C_{ij} - \frac{C_{i3}C_{j3}}{C_{33}}\right] \quad (i,j=1,2), \end{aligned} \tag{34}$$

When Equations 30c, 30f through 30h, and 33 are introduced into the equilibrium Equations 28a through 28e, we obtain two decoupled sets of second-order partial differential equations of equilibrium in terms of five weighted-average displacements:

<u>Fourth-order stretching</u>

$$\left.\begin{aligned} A_{11}u,_{xx} + A_{12}v,_{xy} + A_{66}(u,_{yy} + v,_{xy}) + \frac{C_{13}}{C_{33}}(hq_2),_x &= 0 \\ A_{12}u,_{xy} + A_{22}v,_{yy} + A_{66}(u,_{xy} + v,_{xx}) + \frac{C_{23}}{C_{33}}(hq_2),_y &= 0 \end{aligned}\right] \tag{35a}$$

Sixth-order bending

$$\left.\begin{aligned}
&D_{11}\theta_{y,xx} + D_{12}\theta_{x,xy} + D_{66}(\theta_{x,xy} + \theta_{y,yy}) - G_{55}(w_{,x} + \theta_y) + \frac{2C_{13}}{5C_{33}}(q_1 h^2)_{,x} = 0 \\
&D_{12}\theta_{y,xy} + D_{22}\theta_{x,yy} + D_{66}(\theta_{x,xx} + \theta_{y,xy}) - G_{44}(w_{,y} + \theta_x) + \frac{2C_{23}}{5C_{33}}(q_1 h^2)_{,y} = 0 \\
&G_{55}(w_{,xx} + \theta_{y,x}) + G_{44}(w_{,yy} + \theta_{x,y}) + q_1 = 0
\end{aligned}\right] \tag{35b}$$

with these equilibrium equations being subject to the natural boundary conditions, Equation 29.

It is quite remarkable that comparing these results to Reissner's theory[2-4] (derived for the case of transverse bending of isotropic plates, i.e., excluding the inplane deformations u and v), we find that the equilibrium Equation 35b, the corresponding boundary conditions, Equation 29, and the stress resultants, Equations 33b, and 30f through 30h match those of the Reissner theory.

Computing stresses from the constitutive relations, Equation 12, we find a cubic transverse normal stress

$$\sigma_z = \frac{1}{4}\left[q_1\xi(3 - \xi^2) + 2q_2\right] \tag{36}$$

satisfying the prescribed normal pressure boundary conditions of Equations 1 and 2. From Equation 12, we also determine the parabolic shear stresses of the form

$$\begin{aligned}
\tau_{xz} &= \frac{3k^2}{2} C_{44}(1 - \xi^2)(w_{,x} + \theta_y) = \frac{3}{4h} Q_x(1 - \xi^2) \\
\tau_{yz} &= \frac{3k^2}{2} C_{55}(1 - \xi^2)(w_{,y} + \theta_x) = \frac{3}{4h} Q_y(1 - \xi^2) \, .
\end{aligned} \tag{37}$$

These kinematically admissible transverse stresses (i.e., obtained from constitutive equations) can be shown to satisfy the transverse equilibrium equation of elasticity, Equation 19, and, hence, they are also statically admissible. The transverse stresses can also be seen coincident with those of Reissner,[2-4] who assumed the parabolic transverse shear stresses at the outset, and then computed the transverse normal stress from Equation 19. As far as the inplane stresses, the present τ_{xy} is identical to Reissner's, however, σ_x and σ_y are cubic across the thickness due to ε_z, Equation 23, whereas in Reissner's theory they are linear.

Note that although in statics the equations of equilibrium are conveniently decoupled, in dynamics, all seven equations of motion, Equation 28, are fully coupled. In the latter case, the solution procedure involves substitution of the stress resultants Equation 30 into Equation 28, and simultaneously solving the resulting seven differential equations for the kinematic unknowns.

ASSESSMENT OF THEORY

To qualitatively evaluate the present plate theory, an analytic solution is obtained for the problem of an infinite plate in equilibrium loaded at the top surface by a sinusoidal pressure of the form:

$$q^{+} = q_0 \sin(\pi x/L) \quad \text{on } S^{+} \qquad (q^{-} = 0 \text{ on } S^{-}) \tag{38}$$

where L is a half wavelength of the loading pattern. This problem, having an exact elasticity *solution*,[21] has been widely employed as a benchmark in assessing plate bending theories.

Introducing Equation 38 into Equation 35, and seeking the unknown kinematic variables in the form of $\sin(\pi x/L)$ and $\cos(\pi x/L)$, a system of ordinary differential equations is obtained and readily solved for the unknowns. The solutions are then simplified for the isotropic case by replacing C_{ij}s from Equation 12 with the appropriate expressions in terms of the Lame' constants, λ and μ, which result in the form:

$$v(x) = \theta_x(x) = 0$$

$$u(x) = \frac{q_0 L \lambda}{8\pi\mu(\lambda+\mu)} \cos(\pi x/L)$$

$$\theta_y(x) = -\frac{5 q_0 L^3}{16\pi^3 h^3} \frac{\lambda[1 + 2\mu/\lambda - (2/5)(\pi h/L)^2]}{\mu(\lambda+\mu)} \cos(\pi x/L) \tag{39}$$

$$w(x) = \frac{3 q_0 L^4}{8\pi^4 h^3} \frac{\lambda[1 + 2\mu/\lambda + (2/5)(\pi h/L)^2(3 + 4\mu/\lambda)]}{\mu(\lambda+\mu)} \sin(\pi x/L) .$$

Relations, Equation 32, are then invoked to determine the remaining kinematic variables:

$$w_1(x) = \frac{q_0 h(\lambda+2\mu)}{8\mu(\lambda+\mu)} \sin(\pi x/L)$$

$$w_2(x) = -\frac{q_0 L^2}{86\pi^2 h} \frac{\lambda^2[1 + 2\mu/\lambda - (2/5)(\pi h/L)^2(1 - 85\mu/21\lambda)]}{\mu(\lambda+\mu)(\lambda+2\mu)} \sin(\pi x/L) . \tag{40}$$

The midplane transverse displacement is obtained from Equation 4:

$$u_z(x,\xi=0) = w(x) - 0.3\, w_2(x) \tag{41}$$

Hooke's law, Equation 12, then provides a means for obtaining stresses:

$$\sigma_x(x,\xi) = \frac{q_0 \xi}{4} \left[6(L/\pi h)^2 + \frac{\lambda}{\lambda + 2\mu} \left(\frac{3}{5} - \xi^2 \right) \right] \sin(\pi x/L)$$

$$\sigma_z(x,\xi) = \frac{q_0}{4} (2-\xi)(\xi+1)^2 \sin(\pi x/L) \tag{42}$$

$$\tau_{xz}(x,\xi) = \frac{3 q_0 L}{4\pi h} (1 - \xi^2) \cos(\pi x/L)$$

Several observations can be made regarding these results:

(a) As expected, all weighted-average kinematic variables are identical to those of the Reissner theory. One can, therefore, regard the expansion for u_z as hierarchical, in which Reissner's displacement w is the leading term, and w_1 and w_2 are the higher-order terms allowing a parabolic variation of u_z across the plate thickness.

(b) In the axial stress component σ_x, the first term (linear in ξ) is the Reissner (or classical) stress, while the second term is a higher-order contribution varying as $P_3(\xi)$. Note that the leading stress, of the order $(L/h)^2$, strongly dominates with the decreasing plate thickness.

(c) As established earlier, τ_{xz} and σ_z are the same as in the Reissner theory; they satisfy both constitutive equations and the transverse equilibrium equation of elasticity.

In Figure 1, a normalized value of the maximum midplane transverse displacement is plotted versus the 2h/L ratio, where the results of five bending theories are compared with the exact solution. It is seen that the present and Essenburg's[13] solutions agree quite closely, and they begin to deviate from the exact solution at 2h/L = 1. This is anticipated since the two theories use the same order of displacement approximations, the major difference being that in Essenberg's beam theory, both stresses and displacements are assumed independently. The eleven-variable theory of Lo et al.[15] also appears to be accurate up to the same ratio value. These esults indicate that when a characteristic length of loading (or plate span) is of the order of the plate thickness, application of a higher-order theory is particularly warranted. Similar observations have been made with regard to contact problems,[13] as well as those involving disturbances of geometry such as holes and cutouts, having the characteristic length of the order of the plate thickness. Note that even though Reissner's weighted-average displacement, w, is also shown in the comparison (the only transverse displacement variable in Reissner theory), it does not represent the midplane displacement. In contrast, w is the leading term in the expansion of the present theory, where the complete through-the-thickness distribution is expressed by u_z in Equation 4.

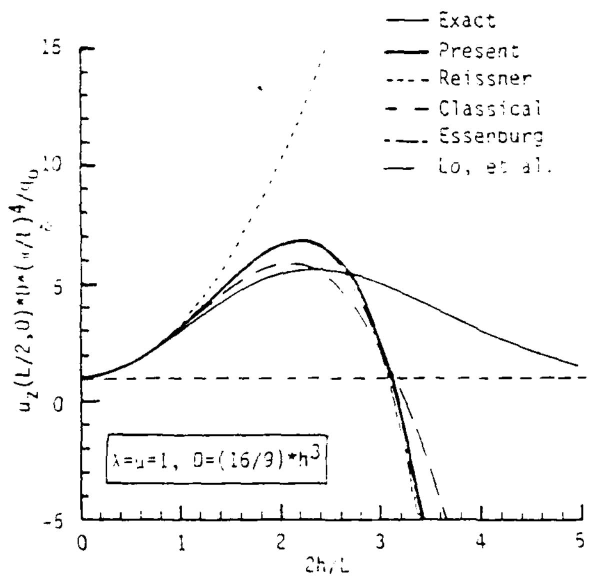

Figure 1. Maximum midplane transverse displacement

In Figures 2 through 4, the present results for σ_x, τ_{xz}, and σ_z are presented for the case of $2h/L = 0.5$, and compared with the corresponding exact stresses. It is seen that the integrity of all stresses is excellent, and particularly the transverse shear and normal stresses for which other displacement theories produce rather inadequate stresses violating stress boundary conditions on the top and bottom plate faces (e.g., References 12 and 15).

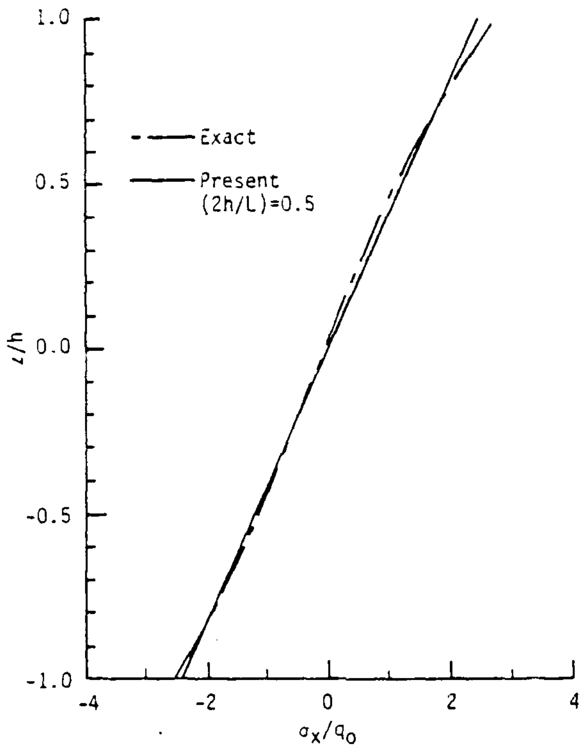

Figure 2. Distribution of maximum σ_x stress across thickness.

CONCLUDING SUMMARY AND FINITE ELEMENT SUITABILITY

A tenth-order displacement theory for orthotropic, homogeneous elastic plates which includes the inplane, transverse shear, and transverse normal deformations was developed. The theory embodies two more kinematic variables than the Reissner and Mindlin shear-deformable theories. A novel means for removing field inconsistency in the transverse shear strains was implemented and led to parabolic strain distributions, satisfying traction-free boundary conditions. In a similar fashion, the transverse normal strain and stress approximations were raised to cubic thickness variations. The virtual work principle yielded seven equations of motion together with exclusively Poisson-type boundary conditions. In the case of static equilibrium, the theory revealed a close relationship to Reissner's theory. An analytic solution to an infinite isotropic plate under a sinusoidal load revealed the high order of accuracy attainable with this theory for all kinematic and stress quantities and, particularly, the transverse stresses for which previous displacement theories yielded inaccurate results.

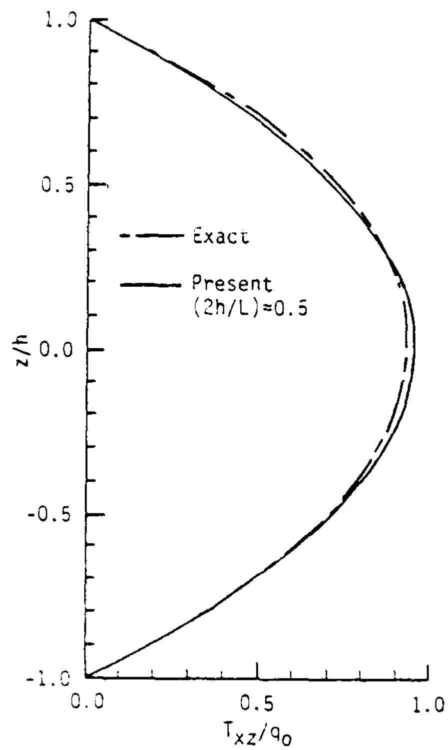

Figure 3. Distribution of maximum τ_{xz} stress across thickness.

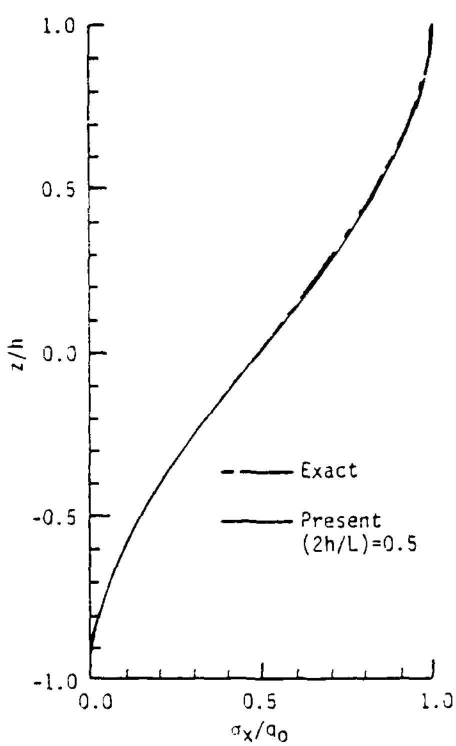

Figure 4. Distribution of maximum σ_x stress across thickness.

Finally, the proposed displacement theory appears to be ideally suited for finite element approximation. The five weighted-average kinematic variables require only C^0-continuity across element boundaries, having the highest spatial derivative in Equation 25b of order one. The remaining two displacements, w_1 and w_2, do not possess derivitives in the energy expressions, thus permitting interelement discontinuous shape functions (i.e., C^{-1} continuous). The latter aspect allows the static condensation of degrees of freedom associated with w_1 and w_2 at the element level. The implication is that computationally efficient elements can be developed by directly adopting successful approaches of shear-deformable elements.[8,19,20] Efforts are currently underway to develop beam and plate elements based on this higher-order displacement theory.

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REFERENCES

1. MINDLIN, R. D. *Influence of Rotatory Inertia and Shear on Flexural Motions of Isotropic, Elastic Plates.* Journal of Applied Mechanics, v. 18, 1951, p. 31-38.
2. REISSNER, E. *On the Theory of Bending of Elastic Plates.* Journal of Mathematics and Physics, v. 23, 1944, p. 184-191.
3. REISSNER, E. *The Effects of Transverse Shear Deformation on the Bending of Elastic Plates.* Journal of Applied Mechanics, v. 12, 1945, p. 69-77.
4. REISSNER, E. *On a Variational Theorem in Elasticity.* Journal of Mathematics and Physics, v. 28-29, 1950, p. 90-95.
5. DONG, S. B., and TSO, F. K. W. *On a Laminated Orthotropic Shell Theory Including Transverse Shear Deformation.* Journal of Applied Mechanics, v. 39, 1972, p. 1091-1097.
6. LEVINSON, M. *An Accurate, Simple Theory for the Statics and Dynamics of Elastic Plates.* Mech. Res. Commun., v. 7, 1980, p. 343-350.
7. REDDY, J. N. *A Simple Higher-Order Theory for Laminated Composite Plates.* Journal of Applied Mechanics, v. 45, 1984, p. 745-752.
8. HUGHES, T. J. R. The Finite Element Method: Linear Static and Dynamic Finite Element Analysis. Chapter 5, Prentice-Hall, Englewood Cliffs, NJ, 1987.
9. HILDEBRAND, F. B., REISSNER, E., and THOMAS, G. B. *Notes on the Foundations of the Theory of Small Displacements of Orthotropic Shells.* NACA T. N. no. 1833, 1949.
10. NAGHDI, P. M. *On the Theory of Thin Elastic Shells.* Quarterly of Applied Mathematics, v. 14, 1957, p. 369-380.
11. NELSON, R. B., and LORCH, D. R. *A Refined Theory of Laminated Orthotropic Plates.* Journal of Applied Mechanics, v. 41, 1974, p. 177-183.
12. WHITNEY, J. M., and SUN, C. T. *A Refined Theory for Laminated Anisotropic, Cylindrical Shells.* Journal of Applied Mechanics, v. 41, 1974, p. 471-476.
13. ESSENBURG, F. *On the Significance of the Inclusion of the Effect of Transverse Normal Strain in Problems Involving Beams with Surface Constraints.* Journal of Applied Mechanics, v. 42, 1975, p. 127-132.
14. REISSNER, E. *On Transverse Bending of Plates, Including the Effects of Transverse Shear Deformation.* Int. J. of Solids and Structures, v. 11, 1975, p. 569-573.
15. LO, K. H., CHRISTENSEN, R. M., and WU, E. M. *A Higher-Order Theory of Plate Deformation, Part 1: Homogeneous Plates.* Journal of Applied Mechanics, v. 44, 1977, p. 663-668.
16. RYCHTER, Z. *A Sixth-Order Plate Theory - Derivation and Error Estimates.* Journal of Applied Mechanics, v. 54, 1987, p. 275-279.
17. REISSNER, E. *Reflections on the Theory of Elastic Plates.* Appl. Mech. Review, v. 38, no. 11, November 1985, p. 1453-1463.
18. LO, K. H., CHRISTENSEN, R. M., and WU, E. M. *Stress Solution Determination for Higher-Order Plate Theory.* Int. J. Solids and Structures, v. 14, 1978, p. 655-662.
19. TESSLER, A., and HUGHES, T. J. R. *A Three-Node Mindlin Plate Element with Improved Transverse Shear.* Comput. Meths. Appl. Mech. Engrg., v. 50, 1985, p. 71-101.
20. TESSLER, A. *Shear-Deformable Bending Elements with Penalty Relaxation* in Finite Element Methods for Plate and Shell Structures, Vol. 1: Element Technology, T. J. R. Hughes and E. Hinton, ed., Pineridge Press, Swansea, U. K., 1986, p. 266-290.
21. LITTLE, R. W. Elasticity. Prentice-Hall, Englewood Cliffs, NJ, 1973, p. 109-112.

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Higher order
Shear deformation

A variationally consistent tenth-order displacement theory of stretching and bending of orthotropic elastic plates is proposed which lends itself perfectly to finite element formulations based upon C^0 and C^1-continuous displacement approximations. The deformations due to all strain and stress components are accounted. The theory is derived from three-dimensional elasticity via the principle of virtual work by expanding the displacement components with respect to the thickness coordinate by means of Legendre polynomials, where the transverse displacement is of a special parabolic form while the inplane displacements are linear. The issues of thickness-expansion related inconsistencies in the transverse shear strains and the transverse normal stress are resolved in a rational fashion. The resulting parabolic shear strains incorporate Reissner's shear correction factor, while the transverse normal strain varies cubically across the plate thickness. The variational principle yields seven equations of motion and exclusively Poisson-type edge boundary conditions. A qualitative assessment of the theory is carried out for the problem of static equilibrium involving an infinite plate under a sinusoidal normal pressure. Pertinent issues on the particular suitability of the theory for the development of efficient displacement plate elements are discussed.

U.S. Army Materials Technology Laboratory
Watertown, Massachusetts 02172-0001
A HIGHER-ORDER PLATE THEORY WITH IDEAL
FINITE ELEMENT SUITABILITY - Alexander Tessler

Technical Report MTL TR 89-85, September 1989, 21 pp-
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